Braided quantum symmetries of graph C^*-algebras

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Outline of the talk

- Braided compact quantum groups - definitions and examples
- Graph C^*-algebras and their braided symmetries
- The braided free unitary quantum group
- Braided symmetries of the Cuntz algebra

Braided compact quantum groups - definitions and examples

Let $q \in \mathbb{C}^\times$ be a nonzero complex number.

Definition

Let A be the universal unital C*-algebra generated by α and γ subject to the following relations:

$$\begin{aligned} \alpha^*\alpha + \gamma^*\gamma &= 1, \\ \alpha\alpha^* + |q|^2\gamma^*\gamma &= 1, \\ \gamma\gamma^* &= \gamma^*\gamma, \\ \alpha\gamma &= \bar{q}\gamma\alpha, \\ \alpha\gamma^* &= q\gamma^*\alpha. \end{aligned} \tag{1}$$

If q were real, then the algebra A is exactly the algebra $C(SU_q(2))$ of continuous functions on the quantum group $SU_q(2)$.

In this case, as is well-known, there is a unique unital $*$-homomorphism $\Delta : A \to A \otimes A$ such that

$$\begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \\ \Delta(\gamma) &= \gamma \otimes \alpha + \alpha^* \otimes \gamma, \end{aligned} \tag{2}$$

which, moreover, is coassociative and satisfies cancellation. If q is not real, the right-hand side of (2) do not satisfy the relations (1); thus there is no morphism Δ satisfying (2).

Surprisingly, there is a monoidal structure $\boxtimes_\zeta$ depending on $\zeta = q/\bar{q} \in \mathbb{T}$ such that the right-hand side of (2) indeed satisfy the relations (1) if we replace $\otimes$ by $\boxtimes_\zeta$, implying the existence of a $\Delta : A \to A \boxtimes_\zeta A$:

$$\begin{aligned} \Delta(\alpha) &= \alpha \boxtimes_\zeta \alpha - q\gamma^* \boxtimes_\zeta \gamma, \\ \Delta(\gamma) &= \gamma \boxtimes_\zeta \alpha + \alpha^* \boxtimes_\zeta \gamma. \end{aligned} \tag{3}$$

Braided tensor product can be defined for two G-C*-algebras, where G is a quasitriangular quantum group. In this talk, we shall restrict ourselves to the case where G is the circle group $\mathbb{T}$.

Definition

We define the category $\mathcal{C}^*_{\mathbb{T}}$ of $\mathbb{T}$-C*-algebras and $\mathbb{T}$-equivariant morphisms as follows. An object of $\mathcal{C}^*_{\mathbb{T}}$ is a pair (X, ρ^X), where X is a unital C*-algebra and $\rho^X \in \mathrm{Mor}(X, X \otimes \mathsf{C}(\mathbb{T}))$ such that

1. $(\rho^X \otimes \mathrm{id}_{\mathsf{C}(\mathbb{T})}) \circ \rho^X = (\mathrm{id}_X \otimes \Delta_{\mathbb{T}}) \circ \rho^X$;
2. $\rho^X(X)(1_X \otimes \mathsf{C}(\mathbb{T})) = X \otimes \mathsf{C}(\mathbb{T})$.

Let (X, ρ^X) and (Y, ρ^Y) be two $\mathbb{T}$-C*-algebras. A morphism $\phi : (X, \rho^X) \to (Y, \rho^Y)$ in $\mathcal{C}^*_{\mathbb{T}}$ (or equivalently, a $\mathbb{T}$-equivariant morphism) is by definition a $\phi \in \mathrm{Mor}(X, Y)$ such that $\rho^Y \circ \phi = (\phi \otimes \mathrm{id}_{\mathsf{C}(\mathbb{T})}) \circ \rho^X$. We write $\mathrm{Mor}^{\mathbb{T}}(X, Y)$ for the set of morphisms between (X, ρ^X) and (Y, ρ^Y) in $\mathcal{C}^*_{\mathbb{T}}$.

Remark

A $\mathbb{T}$-C*-algebra X (with ρ^X understood) comes with an associated $\mathbb{Z}$-grading defined as follows. We call an element $x \in X$ homogeneous of degree $n \in \mathbb{Z}$ if $\rho^X(x) = x \otimes z^n$ and write $\deg(x) = n$. For each $n \in \mathbb{Z}$, we let $X(n)$ denote the set consisting of homogeneous elements of degree n: $X(n) = \{x \in X \mid \deg(x) = n\}$. The collection $\{X(n)\}_{n\in\mathbb{Z}}$ enjoys the following:

1. for each $n \in \mathbb{Z}$, $X(n)$ is a closed subspace of X;
2. for $m, n \in \mathbb{Z}$, $X(m)X(n) \subset X(m+n)$;
3. for each $n \in \mathbb{Z}$, $X(n)^* = X(-n)$;
4. the algebraic direct sum $\bigoplus_{n\in\mathbb{Z}} X(n)$ is norm-dense X.

Let (X, ρ^X) and (Y, ρ^Y) be two objects of $\mathcal{C}^*_{\mathbb{T}}$. Let $\zeta \in \mathbb{T}$ and let $\mathrm{C}(\mathbb{T}^2_\zeta)$ denote the universal unital C*-algebra generated by two unitaries U and V, subject to the relation $VU = \zeta UV$.

Lemma

There exist unique morphisms $\iota_1, \iota_2 \in \mathrm{Mor}(\mathrm{C}(\mathbb{T}), \mathrm{C}(\mathbb{T}^2_\zeta))$ such that $\iota_1(z) = U$ and $\iota_2(z) = V$.

Definition

We define $j_1 \in \mathrm{Mor}(X, X \otimes Y \otimes \mathrm{C}(\mathbb{T}^2_\zeta))$ and $j_2 \in \mathrm{Mor}(Y, X \otimes Y \otimes \mathrm{C}(\mathbb{T}^2_\zeta))$ as the composites

$$X \xrightarrow{\rho^X} X \otimes \mathrm{C}(\mathbb{T}) \xhookrightarrow{\text{imbedding}} X \otimes Y \otimes \mathrm{C}(\mathbb{T}) \xrightarrow{\mathrm{id}_X \otimes \mathrm{id}_Y \otimes \iota_1} X \otimes Y \otimes \mathrm{C}(\mathbb{T}^2_\zeta),$$

and

$$Y \xrightarrow{\rho^Y} Y \otimes \mathrm{C}(\mathbb{T}) \xhookrightarrow{\text{imbedding}} X \otimes Y \otimes \mathrm{C}(\mathbb{T}) \xrightarrow{\mathrm{id}_X \otimes \mathrm{id}_Y \otimes \iota_2} X \otimes Y \otimes \mathrm{C}(\mathbb{T}^2_\zeta),$$

respectively.

Thus, for $x \in X(k)$, $j_1(x) = x \otimes 1_Y \otimes U^k$ and for $y \in Y(l)$, $j_2(y) = 1_X \otimes y \otimes V^l$. It follows that for $x \in X(k)$ and $y \in Y(l)$,

$$j_2(y)j_1(x) = \zeta^{lk} j_1(x)j_2(y) \tag{4}$$

and therefore $j_2(Y)j_1(X) = j_1(X)j_2(Y)$, implying that $j_1(X)j_2(Y)$ is a C*-algebra.

Definition

The braided tensor product $X \boxtimes_\zeta Y$ is defined to be the C*-algebra $j_1(X)j_2(Y)$, i.e., $X \boxtimes_\zeta Y := j_1(X)j_2(Y)$.

Lemma

There is a unique coaction $\rho^{X \boxtimes_\zeta Y}$ of $C(\mathbb{T})$ on $X \boxtimes_\zeta Y$ such that $j_1 \in \mathrm{Mor}^{\mathbb{T}}(X, X \boxtimes_\zeta Y)$ and $j_2 \in \mathrm{Mor}^{\mathbb{T}}(Y, X \boxtimes_\zeta Y)$.

Remark

At the algebraic level, if $x \in X(k)$ and $y \in Y(l)$ then $j_1(x)j_2(y) \in (X \boxtimes_\zeta Y)(k + l)$.

Lemma

Suppose we are given two $\mathbb{T}$*-equivariant morphisms* $\pi_1 \in \mathrm{Mor}^{\mathbb{T}}(X_1, Y_1)$ *and* $\pi_2 \in \mathrm{Mor}^{\mathbb{T}}(X_2, Y_2)$*. Then there is a unique* $\mathbb{T}$*-equivariant morphism* $\pi_1 \boxtimes_\zeta \pi_2 \in \mathrm{Mor}^{\mathbb{T}}(X_1 \boxtimes_\zeta X_2, Y_1 \boxtimes_\zeta Y_2)$ *such that* $(\pi_1 \boxtimes_\zeta \pi_2)(j_1(x_1)j_2(x_2)) = j_1(\pi_1(x_1))j_2(\pi_2(x_2))$*, for* $x_1 \in X_1$ *and* $x_2 \in X_2$*.*

Definition (Meyer+Roy+Woronowicz, 2016)

A braided compact quantum group (over $\mathbb{T}$) G is a triple $G = (\mathrm{C}(G), \rho^{\mathrm{C}(G)}, \Delta_G)$, where $\mathrm{C}(G)$ is a unital C^*-algebra, $\rho^{\mathrm{C}(G)}$ is a $\mathrm{C}(\mathbb{T})$-coaction on $\mathrm{C}(G)$ so that $(\mathrm{C}(G), \rho^{\mathrm{C}(G)})$ is an object of $\mathcal{C}^*_{\mathbb{T}}$, Δ_G is a $\mathbb{T}$-equivariant morphism $\Delta_G \in \mathrm{Mor}^{\mathbb{T}}(\mathrm{C}(G), \mathrm{C}(G) \boxtimes_\zeta \mathrm{C}(G))$ such that

1. $(\Delta_G \boxtimes_\zeta \mathrm{id}_{\mathrm{C}(G)}) \circ \Delta_G = (\mathrm{id}_{\mathrm{C}(G)} \boxtimes_\zeta \Delta_G) \circ \Delta_G$ (coassociativity);
2. $\Delta_G(\mathrm{C}(G))j_2(\mathrm{C}(G)) = \Delta_G(\mathrm{C}(G))j_1(\mathrm{C}(G)) = \mathrm{C}(G) \boxtimes_\zeta \mathrm{C}(G)$ (bisimplifiability).

Theorem (Kasprzak+Meyer+Roy+Woronowicz, 2016)

Let $q \in \mathbb{C}^\times$ and $\zeta = q/\bar{q}$. Recall the algebra A from above; it is the universal unital C^*-algebra generated by α and γ subject to the following relations:

$$\begin{gathered} \alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + |q|^2\gamma^*\gamma = 1, \\ \gamma\gamma^* = \gamma^*\gamma, \quad \alpha\gamma = \bar{q}\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha. \end{gathered} \tag{5}$$

Then the following hold.

- *There is a unique $\mathbb{T}$-action $\rho : A \to A \otimes \mathrm{C}(\mathbb{T})$ such that $\deg(\alpha) = 0$ and $\deg(\gamma) = 1$;*
- *there is a unique $\mathbb{T}$-equivariant unital $*$-homomorphism $\Delta : A \to A \boxtimes_\zeta A$ such that*

$$\begin{aligned} \Delta(\alpha) &= j_1(\alpha)j_2(\alpha) - qj_1(\gamma)^*j_2(\gamma), \\ \Delta(\gamma) &= j_1(\gamma)j_2(\alpha) + j_1(\alpha)^*j_2(\gamma); \end{aligned}$$

- *moreover, Δ is coassociative and bisimplifiable.*

contd.

Definition (Kasprzak+Meyer+Roy+Woronowicz, 2016)

The triple (A, ρ, Δ) constructed in the previous theorem is a braided compact quantum group, called the braided $\mathrm{SU}_q(2)$.

Now, compact quantum groups naturally arise as quantum symmetries of various structures, to name a few:

- of discrete structures;
- of spectral triples;
- of C^*-algebras equipped with orthogonal filtrations;
- of subfactors etc.

Question

Do braided compact quantum groups arise as some sort of "braided symmetries" of objects?

In [Roy, 2021], the author presented examples of braided compact quantum groups as braided symmetries in a precise sense of finite spaces.

Definition (Roy, 2021)

Let $G = (\mathsf{C}(G), \rho^{\mathsf{C}(G)}, \Delta_G)$ be a braided compact quantum group. An action of G (equivalently, a $\mathsf{C}(G)$-coaction) on a $\mathbb{T}$-C*-algebra (B, ρ^B) is a $\mathbb{T}$-equivariant morphism $\eta^B \in \mathrm{Mor}^{\mathbb{T}}(B, B \boxtimes_\zeta \mathsf{C}(G))$ such that

1. $(\mathrm{id}_B \boxtimes_\zeta \Delta_G) \circ \eta^B = (\eta^B \boxtimes_\zeta \mathrm{id}_{\mathsf{C}(G)}) \circ \eta^B$ (coassociativity);
2. $\eta^B(B) j_2(\mathsf{C}(G)) = B \boxtimes_\zeta \mathsf{C}(G)$ (Podleś condition).

Definition (Roy, 2021)

Let (B, ρ^B) be a $\mathbb{T}$-C*-algebra equipped with a G-action $\eta^B \in \mathrm{Mor}^{\mathbb{T}}(B, B \boxtimes_\zeta \mathsf{C}(G))$, where $G = (\mathsf{C}(G), \rho^{\mathsf{C}(G)}, \Delta_G)$ is a braided compact quantum group. A $\mathbb{T}$-equivariant state $f : B \to \mathbb{C}$ on B is one that satisfies

- $(f \otimes \mathrm{id}_{\mathsf{C}(\mathbb{T})}) \rho^B(b) = f(b) 1_{\mathsf{C}(\mathbb{T})}$ for all $b \in B$.

Definition (contd.)

Such an $f: B \to \mathbb{C}$ is said to be preserved under the G-action η^B if

- $(f \boxtimes_\zeta \mathrm{id}_{C(G)})\eta^B(b) = f(b)1_{C(G)}$ for all $b \in B$.

Remark

We recall that $B \boxtimes_\zeta C(G)$ is defined as a sub-C*-algebra of $B \otimes C(G) \otimes C(\mathbb{T}^2_\zeta)$ and clearly $f \otimes \mathrm{id}_{C(G)} \otimes \mathrm{id}_{C(\mathbb{T}^2_\zeta)}$ is defined. We let $f \boxtimes_\zeta \mathrm{id}_{C(G)}$ to be the restriction of $f \otimes \mathrm{id}_{C(G)} \otimes \mathrm{id}_{C(\mathbb{T}^2_\zeta)}$ to $B \boxtimes_\zeta C(G)$.

Definition (Roy, 2021)

A G-action $\eta^B \in \mathrm{Mor}^{\mathbb{T}}(B, B \boxtimes_\zeta C(G))$ of G on a $\mathbb{T}$-C*-algebra (B, ρ^B) is said to be faithful if the $*$-algebra generated by $\{(f \boxtimes_\zeta \mathrm{id}_{C(G)})\eta^B(B) \mid f: B \to \mathbb{C}$ a $\mathbb{T}$-equivariant state$\}$ is norm-dense in $C(G)$.

Graph C*-algebras and their braided symmetries

Definition

Let $E = (E^0, E^1, r, s)$ be a directed graph. The graph C*-algebra C*(E) is the universal C*-algebra generated by families of projections $\{P_v \mid v \in E^0\}$ and partial isometries $\{S_e \mid e \in E^1\}$ subject to the following relations:

1. for $v, w \in E^0$, with $v \neq w$, $P_v P_w = 0$;
2. for $e, f \in E^1$, with $e \neq f$, $S_e^* S_f = 0$;
3. for $e \in E^1$, $S_e^* S_e = P_{r(e)}$;
4. for $e \in E^1$, $S_e S_e^* \leq P_{s(e)}$;
5. for $v \in E^0$ regular, $P_v = \sum_{e \in s^{-1}(v)} S_e S_e^*$.

The C*-algebra C*(E) comes equipped with a $\mathbb{T}$-action, called the gauge action, such that $\deg(P_v) = 0$, for all $v \in E^0$ and $\deg(S_e) = 1$ for all $e \in E^1$.

Definition

Let E be a finite, directed graph without sinks, $\#(E^0) = m$, $\#(E^1) = n$ and $(d_1, \ldots, d_n) \in \mathbb{Z}^n$. The generalized gauge action is the $\mathbb{T}$-action on C*-algebra C$^*(E)$ such that $\deg(P_v) = 0$ for all $v \in E^0$ and $\deg(S_{e_j}) = d_j$, $e_1, \ldots, e_n \in E^1$.

Notations

- Let E be a finite, directed graph without sinks. We denote the vertex matrix (the ij-th entry of which is the number of edges between the i-th and j-th vertices) by D and the spectral radius of the vertex matrix by $\rho(D)$.
- Let us denote the following condition by a $(\dagger)$

$$\rho(D) \text{ is an eigenvalue of } D \text{ such that there is an eigenvector with all entries nonnegative} \qquad (\dagger)$$

and refer to the graphs satisfying it as graphs satisfying condition $(\dagger)$.

We need the following facts from the theory of graph C*-algebras.

Facts

- $C^*(E)$ is the closed linear span of $S_\alpha S_\beta^*$, where α, β are paths in E.
- Let E be a finite, directed graph without sinks satisfying condition (†). Then there is a state τ_E such that

$$\tau_E(S_\alpha S_\beta^*) = \delta_{\alpha\beta} \frac{1}{\rho(D)^{|\alpha|}} \tau_E(P_{r(\alpha)}).$$

  Here $|\alpha|$ and $r(\alpha)$ are the length and the range of the path α, respectively. The symbol $\delta_{\alpha\beta}$ has the obvious meaning: it vanishes when the two paths α and β are different and takes value 1 when α coincides with β.
- The state τ_E is $\mathbb{T}$-equivariant for the generalized gauge action.

Definition (B+Joardar+Roy, 2022)

Let $E = (E^0, E^1, r, s)$ be a finite, directed graph without sinks and let $G = (\mathrm{C}(G), \rho^{\mathrm{C}(G)}, \Delta_G)$ be a braided compact quantum group. A G-action $\eta \in \mathrm{Mor}^{\mathbb{T}}(\mathrm{C}^*(E), \mathrm{C}^*(E) \boxtimes_\zeta \mathrm{C}(G))$ is called linear if there exists $q = (q_{ij})_{1 \leq i,j \leq n} \in M_n(\mathrm{C}(G))$ such that for $1 \leq j \leq n$, we have $\eta(S_j) = \sum_{i=1}^{n} j_1(S_i) j_2(q_{ij})$.

Now we can finally define what we mean by braided quantum symmetries of a graph C^*-algebra.

Definition (B+Joardar+Roy, 2022)

Let $E = (E^0, E^1, r, s)$ be a finite, directed graph without sinks satisfying condition (†) and let τ_E be the state on $\mathrm{C}^*(E)$, mentioned above. We define the category $\mathcal{C}(E, \tau_E)$ as follows.

1. An object of $\mathcal{C}(E, \tau_E)$ is a pair (G, η), where $G = (\mathrm{C}(G), \rho^{\mathrm{C}(G)}, \Delta_G)$ is a braided compact quantum group, and $\eta \in \mathrm{Mor}^{\mathbb{T}}(\mathrm{C}^*(E), \mathrm{C}^*(E) \boxtimes_\zeta \mathrm{C}(G))$ is a τ_E-preserving, linear faithful action of G on $\mathrm{C}^*(E)$.
2. Let (G_1, η_1) and (G_2, η_2) be two objects in $\mathcal{C}(E, \tau_E)$. A morphism $\phi : (G_1, \eta_1) \to (G_2, \eta_2)$ in $\mathcal{C}(E, \tau_E)$ is by definition a $\mathbb{T}$-equivariant Hopf $*$-homomorphism $\phi : \mathrm{C}(G_2) \to \mathrm{C}(G_1)$ such that $(\mathrm{id}_{\mathrm{C}^*(E)} \boxtimes_\zeta \phi) \circ \eta_2 = \eta_1$.

Definition (B+Joardar+Roy, 2022)

A terminal object in $\mathcal{C}(E, \tau_E)$ is called the braided quantum symmetry group of the graph C*-algebra C$^*(E)$ and denoted $(\mathrm{Aut}(\mathrm{C}^*(E)), \eta^E)$.

Theorem (B+Joardar+Roy, 2022)

Let $E = (E^0, E^1, r, s)$ be a finite, directed graph without sinks that satisfies the condition $(\dagger)$. Then $(\mathrm{Aut}(\mathrm{C}^*(E)), \eta^E)$ exists.

The braided free unitary quantum group

Definition (B+Joardar+Roy, 2022)

For an admissible $F \in \mathrm{GL}(n, \mathbb{C})$, we define $\mathrm{C}(\mathrm{U}^+_\zeta(F))$ to be the universal unital C^*-algebra with generators u_{ij} for $1 \leq i, j \leq n$ subject to the relations that make u and $F\overline{u}_\zeta F^{-1}$ unitaries, where $u = (u_{ij})_{1\leq i,j\leq n}$ and $\overline{u}_\zeta = (\zeta^{d_i(d_j - d_i)} u^*_{ij})_{1\leq i,j\leq n}$.

Remarks

- It takes some time to say what an "admissible" F is and why only these F are considered.
- The d_i, $i = 1, \ldots, n$, appearing above comes with such an "admissible" F.
- The identity matrix $I_n \in \mathrm{GL}(n, \mathbb{C})$ and in fact, any invertible diagonal matrix is admissible; moreover, these are the only matrices that will be needed to state our results.

Theorem (B+Joardar+Roy, 2022)

The following hold.

- *There is a unique* $\mathbb{T}$*-action*

$$\rho^{\mathsf{C}(\mathsf{U}^+_\zeta(F))} : \mathsf{C}(\mathsf{U}^+_\zeta(F)) \to \mathsf{C}(\mathsf{U}^+_\zeta(F)) \otimes \mathsf{C}(\mathbb{T})$$

  such that $\deg(u_{ij}) = d_j - d_i$ *for* $1 \leq i, j \leq n$*;*

- *there is a unique unital* $*$*-homomorphism*

$$\Delta_{\mathsf{U}^+_\zeta(F)} : \mathsf{C}(\mathsf{U}^+_\zeta(F)) \to \mathsf{C}(\mathsf{U}^+_\zeta(F)) \boxtimes_\zeta \mathsf{C}(\mathsf{U}^+_\zeta(F))$$

  such that $\Delta_{\mathsf{U}^+_\zeta(F)}(u_{ij}) = \sum_{k=1}^{n} j_1(u_{ik}) j_2(u_{kj})$ *for* $1 \leq i, j \leq n$*;*

- *furthermore,* $\Delta_{\mathsf{U}^+_\zeta(F)}$ *is* $\mathbb{T}$*-equivariant, coassociative and bisimplifiable.*

Definition (B+Joardar+Roy, 2022)

The triple $(\mathrm{C}(\mathrm{U}_\zeta^+(F)), \rho^{\mathrm{C}(\mathrm{U}_\zeta^+(F))}, \Delta_{\mathrm{U}_\zeta^+(F)})$, constructed in the previous theorem is a braided compact quantum group, called the braided free unitary quantum group, denoted $\mathrm{U}_\zeta^+(F)$.

Notation

We write $\mathrm{U}_\zeta^+(n)$ instead of $\mathrm{U}_\zeta^+(I_n)$

Sketch of proof

of the main Theorem

The first step is to observe the following.

Proposition

Let $(G,\eta) \in \mathrm{Obj}(\mathcal{C}(E,\tau_E))$ be an object in the category $\mathcal{C}(E,\tau_E)$. Then the C^**-algebra* $\mathrm{C}(G)$ *is a quotient of the* C^**-algebra* $\mathrm{C}(\mathrm{U}_\zeta^+(F))$*, for some diagonal F.*

Definition

We now define another category $\mathcal{C}'(E)$ as follows. An object of $\mathcal{C}'(E)$ is a triple (X,ρ^X,η^X) which consists of

1. a $\mathbb{T}$-C^*-algebra (X,ρ^X) generated by $\{t_{ij}\}_{1\leq i,j\leq n}$ such that the two matrices

$$t=(t_{ij})_{1\leq i,j\leq n},\quad F\bar{t}_\zeta F^{-1}=(\zeta^{d_i(d_j-d_i)}F_{ii}t_{ij}^*F_{jj}^{-1})_{1\leq i,j\leq n},$$

are unitaries.

Definition (contd.)

2. a $\mathbb{T}$-equivariant morphism $\eta^X \in \mathrm{Mor}^{\mathbb{T}}(C^*(E), C^*(E) \boxtimes_\zeta X)$ such that, for each $1 \le j \le n$, $\eta^X(S_j) = \sum_{i=1}^n j_1(S_i)j_2(t_{ij})$.

Let (X, ρ^X, η^X) and (Y, ρ^Y, η^Y) be two objects in the category $\mathcal{C}'(E)$. A morphism $\phi : (X, \rho^X, \eta^X) \to (Y, \rho^Y, \eta^Y)$ in $\mathcal{C}'(E)$ is by definition a $\mathbb{T}$-equivariant morphism $\phi : X \to Y$ such that $(\mathrm{id}_{C^*(E)} \boxtimes_\zeta \phi) \circ \eta^X = \eta^Y$.

Lemma

An initial object in the category $\mathcal{C}'(E)$ exists, denoted $(\mathcal{U}, \rho^{\mathcal{U}}, \eta^{\mathcal{U}})$. The generators are written $[u_{ij}]$, with the $\mathbb{T}$-action such that $\deg([u_{ij}]) = d_j - d_i$. Finally, $\eta^{\mathcal{U}}$ is given by

$$\eta^{\mathcal{U}}(S_j) = \sum_{i=1}^n j_1(S_i)j_2([u_{ij}]), \quad 1 \le j \le n.$$

It turns out that $\mathcal{U}$ can be provided with more structures so as to make it a braided compact quantum group. The first step is to observe that $\mathcal{U} \boxtimes_\zeta \mathcal{U}$ can be made into an object of the category $\mathcal{C}'(E)$.

Lemma

The braided tensor product $\mathcal{U} \boxtimes_\zeta \mathcal{U}$ of $\mathcal{U}$ with itself can be made into an object $(\mathcal{U} \boxtimes_\zeta \mathcal{U}, \rho^{\mathcal{U}\boxtimes_\zeta\mathcal{U}}, \eta^{\mathcal{U}\boxtimes_\zeta\mathcal{U}})$ of $\mathcal{C}'(E)$.

Corollary

There exists a unique $\mathbb{T}$-equivariant unital $*$-homomorphism $\Delta_\mathcal{U} \in \mathrm{Mor}^{\mathbb{T}}(\mathcal{U}, \mathcal{U} \boxtimes_\zeta \mathcal{U})$ such that

$$\Delta_\mathcal{U}([u_{ij}]) = \sum_{k=1}^{n} j_1([u_{ik}]) j_2([u_{kj}]).$$

Furthermore, $\Delta_\mathcal{U}$ is coassociative and bisimplifiable.

Corollary

There exists a braided compact quantum group (over $\mathbb{T}$) G_E such that $(\mathrm{C}(G_E), \rho^{\mathrm{C}(G_E)}, \Delta_{G_E}) = (\mathcal{U}, \rho^{\mathcal{U}}, \Delta_{\mathcal{U}})$. Furthermore, G_E acts linearly, faithfully on $\mathrm{C}^*(E)$ preserving τ_E via $\eta^{\mathcal{U}}$, denoted henceforth by η^{G_E}.

Theorem

$(G_E, \eta^{G_E}) \cong (\mathrm{Aut}(\mathrm{C}^*(E)), \eta^E)$.

Braided symmetries of the Cuntz algebra

We recall that the Cuntz algebra $\mathcal{O}_n$ is the graph C*-algebra corresponding to the graph (denoted by $E_{\mathcal{O}_n}$) with a single vertex and n-loops at it. Explicitly, $\mathcal{O}_n$ is the universal unital C*-algebra generated by S_i for $1 \leq i \leq n$ subject to the relations

$$S_i^* S_j = \delta_{ij} \quad (1 \leq i, j \leq n), \text{ and } S_1 S_1^* + \cdots + S_n S_n^* = 1.$$

$\mathcal{O}_n$ is equipped with the generalized gauge action $\rho^{\mathcal{O}_n} : \mathcal{O}_n \to \mathcal{O}_n \otimes \mathrm{C}(\mathbb{T})$ given by $\deg(S_i) = d_i$, $1 \leq i \leq n$, and $(d_1, \ldots, d_n) \in \mathbb{Z}^n$.

Proposition (B+Joardar+Roy, 2022)

There is a unique unital $*$-homomorphism $\eta : \mathcal{O}_n \to \mathcal{O}_n \boxtimes_\zeta \mathsf{C}(\mathsf{U}_\zeta^+(n))$ such that $\eta(S_j) = \sum_{i=1}^n j_1(S_i) j_2(u_{ij})$ for $1 \leq i, j \leq n$. Furthermore, η is $\mathbb{T}$-equivariant, coassociative and satisfies Podleś condition.

Theorem (B+Joardar+Roy, 2022)

$(\mathrm{Aut}(\mathcal{O}_n), \eta^{\mathcal{O}_n}) \cong (\mathsf{U}_\zeta^+(n), \eta)$.

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Thank you!